Exercise 6

Use the *Laplace transform method* to solve the Volterra integral equations:

$$u(x) = \cos x - \sin x + 2\int_0^x \cos(x-t)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\begin{aligned}
\mathcal{L}\{u(x)\} &= \mathcal{L}\left\{\cos x - \sin x + 2\int_0^x \cos(x-t)u(t)\,dt\right\} \\
U(s) &= \mathcal{L}\{\cos x\} - \mathcal{L}\{\sin x\} + 2\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\} \\
&= \mathcal{L}\{\cos x\} - \mathcal{L}\{\sin x\} + 2\mathcal{L}\{\cos x\}U(s) \\
&= \frac{s}{s^2+1} - \frac{1}{s^2+1} + 2\left(\frac{s}{s^2+1}\right)U(s)
\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}
\left(1 - \frac{2s}{s^2+1}\right)U(s) &= \frac{s}{s^2+1} - \frac{1}{s^2+1} \\
U(s) &= \frac{\frac{s}{s^2+1} - \frac{1}{s^2+1}}{1 - \frac{2s}{s^2+1}} \\
&= \frac{s-1}{(s^2+1) - 2s} \\
&= \frac{s-1}{(s-1)^2} \\
&= \frac{1}{s-1}
\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}
u(x) &= \mathcal{L}^{-1}\{U(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\
&= e^x
\end{aligned}$$